ρ_f = density of the fluid
ρ_p = density of the packing particles

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A Correlation of the Frictional Characteristics for Turbulent Flow of Dilute Viscoelastic Non-Newtonian Fluids in Pipes

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A formula has been determined which satisfactorily represents, for the existing data, the frictional characteristics of the turbulent flow of a dilute viscoelastic non-Newtonian fluid in a pipe. This formula contains two elastic fluid parameters, one of which is strongly dependent on both polymer solute and concentration and the other appears to be a constant and independent of the polymer solutes which were used in this report. A rheometer is proposed based on this formula which should be useful in classifying fluids of this type.

The purpose of this paper is to propose a formula which, based upon the limited data available, shows promise in predicting the frictional characteristics of turbulent flow of dilute viscoelastic non-Newtonian fluids in pipes. Since no adequate theoretical procedure exists for the evaluation of turbulent boundary-layer flows, the derivation of the above mentioned formula was, by necessity, empirical. Nevertheless, some of the empirical relationships used imply the presence of a basic underlying mechanism, the understanding of which could lead to the formulation of a theoretical procedure.

In the initial work done at this laboratory by Wells (*1*), who used guar gum (J-2P, product of the Western Company) solutions of concentrations from 500 to 4,000 p.p.m. in water, there was a strong implication that the viscoelastic effect of a non-Newtonian fluid on turbulent flow caused a decrease in the Prandtl mixing length constant. A similar conclusion was reached by Elata† and Tirosh in a recent publication (*2*) in which they used very dilute concentrations (50 to 400 p.p.m.) of guar gum ("Jaguar," product of Stein and Hall) in water. The work of Ernst (*3*), however, very conclusively showed that, for a dilute concentration (500 p.p.m.) of CMC 7HSP in water, the mixing length constant in the turbulent portion of the flow was not changed; and there appeared to be a thickening of the laminar and buffer layers of the flow near the wall. In the light of these results, the work of Wells and of Elata and Tirosh were re-evaluated and it was found that, at least for dilute concentrations of viscoelastic fluids, they could be interpreted to show the value of the mixing length constant to be unchanged. It is the evaluation and interpretation of the results of these three reports which form the basis of this paper.

EFFECTS OF ELASTICITY ON THE UNIVERSAL VELOCITY PROFILE

The data in the paper of Ernst were obtained from the measured flow quantities and pressure drops in two pipe sizes (0.650 and 1.427 in. I.D.) as well as detailed velocity surveys across the pipes. The latter were presented in the form of the universal logarithmic velocity

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† As a result of work subsequent to reference 2, Dr. Elata, in a private communication with the author, agreed that the mixing length constant does not appear to change and that a velocity profile shift does occur.

profile; the turbulent portion of which can be expressed mathematically (*4, 5*) as

$$u/u_* = A \log (y u_*)/(\nu_w) + B \qquad (1)$$

Equation (1) has been shown to be valid for the turbulent portion of the flow for Newtonian fluids in pipes; in which case

$$A = 2.303/k = 5.77$$

$$B = B_N = 5.5$$

$$k = 0.4, \text{ Prandtl mixing length constant}$$

From the velocity profile plots in Ernst's report, it can be seen that the slope had not changed, that is, A of Equation (1), and therefore, the mixing length constant was the same as for Newtonian flows. The obvious non-Newtonian effect on the flow was to increase the constant B and to make it a variable which increased with increasing Reynolds number and decreasing pipe diameter. This effect was also present in the work of Wells for the lowest concentration solution.

From these results, one is tempted to speculate that the turbulent portion of the flow is essentially unaffected by the viscoelastic properties, since the mixing length distribution within the flow was unchanged. In this event, it would be logical to assume that the laminar sublayer, due to the elastic fluid properties, has been made less sensitive to disturbances being impressed upon it from above and thus becomes thicker. In the three reports cited above, the purely viscous properties of the fluids used were either Newtonian or very slightly non-Newtonian; therefore it is presumed that any deviation from Newtonian flow characteristics must be due to elastic effects. As a consequence of the above argument, it was concluded that any dampening within the laminar sublayer due to elastic effects (with the resulting increase in sublayer thickness) must be a function of the shear stress or shear rate within the sublayer, since these are the only flow parameters which could reasonably be expected to affect the polymeric molecules within the fluid. As a first trial, the friction velocity u_*, which is related to the shear stress, was chosen as a convenient representation of the flow parameter within the sublayer.

Figure 1 shows a plot of B vs. u_*, which was determined from the velocity profile data of Ernst. The oblique line was faired through the small pipe data, since it was considered the most reliable. The lower end of this line stops at a B of 5.5 which is the value for Newtonian flow. This results in a critical value of u_* equal to 0.23 ft./sec. below which the flow properties are Newtonian—typical for this type of fluid (*1* to *3, 6, 8*). As mentioned in reference *3*, a deterioration of the alclad coating of the large pipe resulted in corrosion occurring on its internal surface which apparently affected the pressure drop data. As a result, the large pipe data were not very consistent; however, they do tend to scatter about the line through the small pipe data. Actually, if the argument presented above is valid, data presented on this type of plot should correlate and be independent of pipe diameter. As part of a larger program, these data are currently being checked with stainless steel pipes.

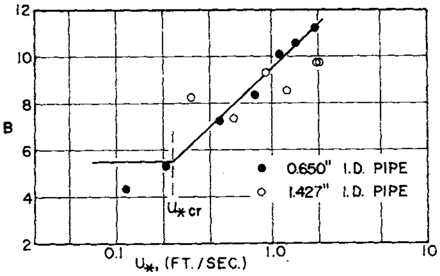

Fig. 1. Effect of wall shear stress on velocity profile shift for 0.05% solution of CMC in water (3).

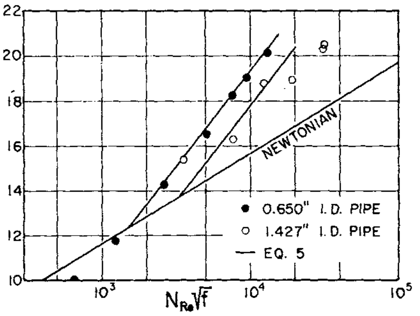

Fig. 2. Comparison between measured and semiempirical pipe flow characteristics for 0.05% solution of CMC in water (3).

DERIVATION OF FRICTION FACTOR—REYNOLDS NUMBER FORMULA

Pipe flow data are customarily presented in the form of friction factor vs. Reynolds number. The non-Newtonian effects on this relationship can be derived by the use of the information presented in Figure 1 and by the use of Equation (1). Equation (1) evaluated at the center of the pipe becomes

$$(u_m)/(u_*) = A \log (D u_*)/(2\,\nu_w) + B \qquad (2)$$

The value of B in Equation (2) can be found from Figure 1 and can be conveniently written as

$$B = B_N + \alpha \log (u_*)/(u_{*cr}); \quad u_* \geqq u_{*cr} \qquad (3)$$

Since the friction factor and Reynolds number are usually written in terms of the average velocity rather than the maximum, the following relationship for Newtonian flow from reference *4* or *5* is needed:

$$\overline{u} = u_m - 4.07\, u_* \qquad (4)$$

A check of the data from reference 3 showed this to be valid also for the flow of dilute non-Newtonian fluids.

Equations (2), (3), and (4) can be combined to give the following equation:

$$\frac{1}{\sqrt{f}} = \left(4 + \frac{\alpha}{\sqrt{2}}\right) \log N_{Re}\sqrt{f} - 0.394 - \frac{\alpha}{\sqrt{2}} \log \frac{\sqrt{2}\, D\, u_{*cr}}{\nu_w} \qquad (5)$$

$$u_* \geqq u_{*cr}$$

In this equation, the values of A and B_N were changed slightly from the values given previously to $A = 5.66$ and $B_N = 6.07$. This is consistent with the approach used for Newtonian fluids in order that the equation derived from the velocity profile would have a better fit with the measured pressure drop data (see references *4* and *5*). Equation (5) reduces to the standard equation for Newtonian fluids upon setting α equal to zero.

Equation (5) states that if turbulent pipe flow data are plotted as $1/\sqrt{f}$ vs. $\log N_{Re}\sqrt{f}$, a straight line should result. This was done for the data of reference 3 and the results are shown in Figure 2. The symbols represent the data found by measuring the pressure drop and average flow in the pipe. The lines represent Equation (5) with $\alpha = 6.3$ as evaluated from Figure 1. Thus it can be seen from Figure 2 that the procedure which resulted in Equa-

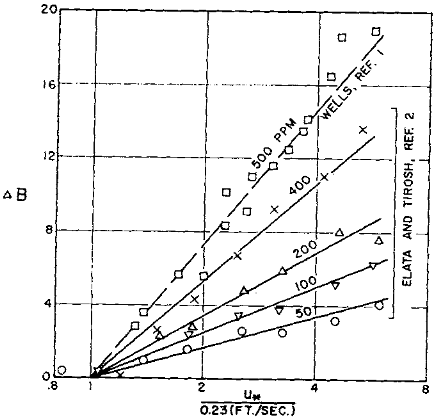

Fig. 3. Correlation of data taken from references *1* and *2* for various concentrations of guar gum type of additives in water.

tion (5) gives a good correlation between the data evaluated from the velocity profiles and that evaluated from the bulk flow properties for the small pipe. Again the large pipe data scatter about the line and are not nearly as consistent as the small pipe data.

EVALUATION OF THE RESULTS OF WELLS AND ELATA AND TIROSH

The question naturally arises as to how well other data fit the formula given by Equation (5). Of particular interest is the data of Elata and Tirosh as given in Figure 7 of reference *2*. For the reader's reference, this figure is essentially the same as Figure 2 of the present report. Data were presented for four different concentrations of the polymer which resulted in four straight lines whose slopes increased with increasing concentration, and all of the curves intersected the Newtonian line at a value of $N_{Re}\sqrt{f}$ of approximately 9,000. It should be noted that the friction factor f of the present report differs by a factor of 4 from that of reference *2*. It can be seen that the form of Equation (5) fits these data, since α would change with concentration and all curves would meet the Newtonian curve when

$$N_{Re}\sqrt{f} = \frac{\sqrt{2}\,u_{*cr}\,D}{\nu_w}$$

In order to have a common basis of comparison, all the data were reduced to the form of Figure 1. This was accomplished by evaluating B with Equations (2) and (4) to achieve the following equation:

$$B - 6.07 = \Delta B = \bar{u}/u_* - 2 - 5.66 \log (Du_*)/(2\nu_w) \quad (6)$$

where ΔB represents the increase in the local velocity ratio u/u_* of the non-Newtonian over the Newtonian flows. Again the constants were changed slightly following the procedure used in Newtonian flows so that the velocity data and pressure drop data are consistent.

The data of reference *2* reduced in this manner are shown in Figure 3. As expected, the curves of constant concentration appear as straight lines; however, as was not expected, the flows became Newtonian at the same critical friction velocity as determined from the data of reference 3. It therefore can be seen that the results of Elata and Tirosh can be explained by a thickening of the sublayer while maintaining the Prandtl mixing length parameter unchanged from its Newtonian value. The data for a fluid concentration of 500 p.p.m. from the work of Wells taken in the 0.65-in. I.D. pipe are also given in this figure; again, it is seen to be consistent with the other data.

DISCUSSION

The lack of enough of the proper type of experimental data makes it very difficult to evaluate effectively the validity of the method and ideas proposed in this paper. Within this limitation, Equation (5) appears to describe adequately the frictional characteristics of the flow of a dilute elastic non-Newtonian fluid whose purely viscous properties are Newtonian. It does this with, at most, two elastic fluid parameters, α and u_{*cr}, and if future work shows that u_{*cr} is constant for solutions of practical interest, the problem would be reduced to the evaluation of a single elastic fluid property.

The fact that the critical shear stress was constant for all of the fluids and test conditions represented by the data used in this paper was somewhat surprising. Originally it was thought that this quantity would be quite sensitive to the type of non-Newtonian additives used in the experiment. The mechanism envisioned was that at a critical shear stress the forces acting on the polymeric molecules would become large enough to distort them in some fashion so as to store energy elastically. Thus, under fluctuations, this process would, through the storage of energy, decrease the tendency of the laminar flow in the sublayer to become turbulent. The sublayer thickness would then increase with a resulting decrease in the wall shear stress.

The work of Virk, Merrill, Mickley, and Smith (*6*) and of Fabula, Lumley, and Taylor (*7*) show that the critical value of u_* does, in general, vary with the polymer molecule or, more specifically, with the size of the molecule. The results of an experimental investigation presented in reference *6*, with dilute solutions of four different size polymers of polyethylene oxide in water, showed that the square root of the critical wall shear stress was inversely proportional to the radius of gyration of the molecule. Although the value of the critical wall shear stress for the smallest molecule tested was ten times that used in this paper, the difference between the u_{*cr} found by Virk and found in this paper rapidly disappeared as the molecule size was increased. It is also of interest to note that for the original work of Toms (*8*), who used polymethyl methacrylate in monochlorobenzene, the critical wall shear stress, as evaluated in reference *6*, was also essentially the same as found for the solutions used in this paper. At this point, one is tempted to speculate that the critical wall shear stress approaches a value of approximately 0.1 lb./sq. ft. or 50 dynes/sq. cm. (the values found in this paper) for polymeric solutions which are capable of affecting a large drag reduction in turbulent flows. The non-Newtonian effects for the smaller molecule sizes used in reference *6* were quite small. This position is somewhat jeopardized by the results of Giles (*9*) in which the critical wall shear stress, as presented in reference 7, was found to be approximately 5 dynes/sq. cm. for a very large polymeric molecule. Thus more experimental and theoretical work will be necessary before this point can be resolved.

At present there is a need for a method that would effectively and accurately determine the elastic fluid parameters which affect the turbulent flow characteristics for dilute non-Newtonian viscoelastic fluids. The difficulties associated with achieving this goal are well known. They arise primarily because the effects of the elastic fluid parameters are very small as compared with those of the viscous parameters for this class of fluids. However, in the case of turbulent pipe flows, the elastic effects for these fluids appear to be quite large and can be measured with relative ease. Thus it would seem appropriate to suggest that the parameters α and possibly u_{*cr} be considered as elastic fluid parameters and a rheometer, in the form of a turbulent pipe flow facility, could be used to evaluate these constants for various fluids. At the very least, this would be an effective interim method by which elastic non-Newtonian fluids could be classified. On the other hand, an understanding of the underlying mechanism involved could lead to a theoretical basis for α and u_{*cr} which would then make this type of rheometer a basic device. An analysis of this type would be difficult to formulate, since it would involve the determination of the viscoelastic effects on the thickness of the sublayer for a turbulent boundary layer. This would certainly require a procedure for the determination of the sublayer thickness itself which, to date, is not available even for Newtonian fluids. However, with more systematic data available, the entire process will be better understood. It is, therefore, strongly recommended that this type of rheometric device be seriously considered for future classifications of these fluids.

ACKNOWLEDGMENT

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NOTATION

A = $2.303/k$
B = constant in the law-of-the-wall equation
B_N = value of B for Newtonian fluids
ΔB = $B - B_N$
D = I.D. of pipe
f = friction factor, $\tau_w/(\frac{1}{2}\rho\bar{u}^2)$
k = Prandtl mixing length constant equal to 0.4
N_{Re} = Reynolds number, $\bar{u}D/\nu_w$
$\bar{u}$ = mean velocity of flow through the pipe
u_m = maximum flow velocity in the pipe
u_* = friction velocity, $\sqrt{\tau_w/\rho}$
u_{*cr} = critical value of u_*, taken as 0.23 ft./sec. in this report
y = radial distance from the pipe wall

Greek Letters

α = fluid property parameter defined in Equation (3)
ν_w = kinematic viscosity at the wall
ρ = mass density of the fluid
τ_w = shear stress at the wall

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An Experimental Study of Falling Liquid Films

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A new method for measuring wavelength and wave velocity is described, and experimental values for water flowing down a vertical plane are compared with a numerical solution of the Orr-Sommerfeld equation. Good agreement is obtained in the region near the top of the film where small disturbance theory is expected to be valid. Experimental Reynolds numbers ranged from 8 to 120.

There are a wide variety of heat and mass transfer processes which involve transport across a thin liquid film. The current list ranges from ablating space vehicles (*17*) to film cooling of turbine blades (*12*) to the typical chemical engineering problems of gas absorption in wetted-wall towers and condensation in cooler condensers. A comprehensive survey of current theories and experimental data for both laminar and turbulent films has recently been presented by Fulford (*7, 8*) and the usual survey will not be incorporated here.

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